Master Project

Numerical Integration - An Introduction for the Boundary Element Method

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1 Preliminaries

1.1 Introduction

While the Finite Element Method (FEM) is known to almost every engineer, the Boundary Element Method (BEM) is rather unknown. The BEM has advantages to the FEM in certain application fields. Therefore, every engineer involved in numerical calculations should have at least basic knowledge on this subject. The most remarkable difference between the BEM and the FEM is the discretisation. While in the FEM the whole domain has to be described with a finite element mesh, in the BEM usually only the boundary of a domain has to be discretised into a finite number of boundary elements (see Figure 1.1). But there are more significant differences in the calculation of the domain results. In the FEM, the unknowns are the node displacements. After the solution for the displacements is obtained, the strains and stresses are calculated from the node displacements of each element. In the BEM, unknown boundary displacements and unknown boundary tractions are calculated by solving a single system of equations. Domain results are computed by using a representation formula, e.g. Somigliana’s Identity. Fundamental solutions are used in the BEM (see Chapter 3). These fundamental solutions can contain singularities, which make them hard to integrate. Therefore, special integration techniques have to be used. A common method to solve weakly singular integrals for the BEM is reviewed in this work.

Looking at Figure 1.1 it might not be obvious, that the dimension of BEM problems is reduced by one order in respect of the FEM. While the FEM uses two-dimensional elements (areas) to discretise two-dimensional problems, the BEM uses one-dimensional elements (lines). Still referring to the example in Figure 1.1, the solution of the FEM gives jumps of strains and stresses on the border between two elements. This is a violation of the condition of compatibility. The BEM gives continuous stresses and strains in the domain.

![Figure 1.1: Discretisation for a two-dimensional problem](a) FEM discretisation (b) BEM discretisation

1.2 Basic Terms

This section gives a short review of the most important terms used in this work.
1 Preliminaries

1.2.1 Differential Equation

A differential equation (DE) includes a function and one or more derivates of this function at once. An example for engineers is the equation for the deflection curve of a uniformly loaded beam.

\[ w''''(x) = \frac{q(x)}{EI} . \tag{1.1} \]

This equation refers the forces to the deformations and vice versa.

1.2.2 Domain

A domain describes a finite or infinite continuous region. The simplest example is a domain infinite to all directions. As seen later, this domain is used in the calculation of the fundamental solutions. Another example is a cavern or a tunnel. Here, the transition from the domain outside to the domain inside is defined by the boundary elements, as it can be seen in Figure 1.3. The domain itself is "outside" of the cavern. Again we deal with an infinite domain. Figure 1.4 is an example for a finite domain, describing a cantilever beam.

1.2.3 Boundary Condition

The boundary conditions (BC) define forces or deformations on some parts of a domain \( \Omega \) (see Figure 1.4). Using a cantilever beam as a simple example (as shown in Figure 1.2), the deformation...
and the rotation have to be zero at the fixed end and the moment has to be equal to zero at the free end. Geometrical BCs are called Dirichlet conditions. BCs defining forces or stresses are called Neumann conditions.

1.2.4 Jacobian Determinant

The Jacobian determinant \( J \), often just called Jacobian, is the determinant of the Jacobian Matrix. The Jacobian Matrix includes the first-order partial derivatives of a vector function \([1]\).

1.2.5 Shape Function

Elements usually are defined by nodes and interpolation functions. Linear elements have fewer nodes than quadratic elements. For instance a linear boundary element in a two-dimensional problem is a line defined by two nodes, one on each end (see Figure 1.5(a)). A quadratic element in the same problem has three nodes. Node 1 and 2 on the ends and node 3 in the middle of the line (see Figure 1.5(b)).

Now, shape functions are used to interpolate the quantities, e.g. displacements and tractions, for an arbitrary point on the element. Shape functions of linear elements are linear functions, those of quadratic elements are quadratic. \([7]\)

1.2.6 Intrinsic Coordinates

The elements used in the BEM can have arbitrary shapes. They are defined by nodes, which are defined in the global coordinate system. Those facts make it difficult to treat them in numerical methods. Therefore, the shape is defined by standard shape functions and the global coordinates are transformed to intrinsic coordinates, also called local coordinates. These standardized elements are much easier to treat as the borders of the element are on ±1 of the intrinsic coordinate system. \([7]\)
1.2.7 Boundary and Domain Integral

A boundary integral is a integral defined over a closed boundary $\Gamma$. In the BEM the integral consists of the product of a fundamental solution times boundary displacements or boundary tractions. The *domain integral* is an enhancement of the boundary integral. It involves a fundamental solution $E_{ijk}$ and an internal variable, e.g. initial stresses, and has to be taken over the domain $\Omega^0$, where the internal variable acts.

For non-linear problems, which involve a domain integral, not only the boundary has to be discretised, but also those parts of the domain, which behave non-linear. For the discretisation of the non-linear domain similar elements as in the FEM are used. For two-dimensional and three-dimensional problems area cells and volume cells are used respectively.\[10\]
2 Numerical Integration

The integrals found in the BEM can barely be solved analytically. The integrands are products of shape functions $N(\xi, \eta)$ and fundamental solutions $U(P, Q)$ or even other fundamental solutions found in Chapter 3 [8, p. 175]. These integrals are quite complicated, as they contain exponentiated sums and fractions. It is nearly impossible to provide antiderivatives for all functions, so the best way is to solve the integrals numerically. Several techniques are presented in the following sections and the results are compared to the analytical solution. The integral considered has the form

$$I = \int_{-1}^{1} \cos\left(\frac{\pi}{2}x\right) \, dx = \frac{4}{\pi} \approx 1.2732.$$  \hspace{1cm} (2.1)

2.1 Rectangular Rule

In this method, the function is approximated with a step function, and the area (the integral) under this step function graph is calculated rectangle by rectangle:

$$I \approx \sum_{i=1}^{n} f(x_{a} + l(i + \frac{1}{2}))$$  \hspace{1cm} (2.2)

By using $n = 10$ and $l = 0.2$, the result for the considered integral is 1.2785. This means an error of 0.4%. This surprisingly accurate result should not be overvalued. For other functions, the rectangular rule can give inaccurate results and the result strongly depends on the number of rectangles. The result can be improved by increasing the number of rectangles. [3]
2.2 Trapezoidal Rule

The trapezoidal rule approximates the function to integrate with linear segments, whose nodes are on the graph of the function. The area under this polygonal chain represents the integral

$$I \approx \frac{x_b - x_a}{n} \left[ \frac{f(x_a) + f(x_b)}{2} + \sum_{k=1}^{n-1} \left( x_a + k \frac{x_b - x_a}{n} \right) \right].$$  \hspace{1cm} (2.3)

Figure 2.2: Numerical integration by trapezoidal rule

This method gives a result of 1.2628. Due to the fact that the graph is convex, this method gives too small results for this integration (see Figure 2.2). The error is 0.8%.

2.3 Simpson’s Rule

Simpson’s rule is well known to engineers. By using this rule, integrals of up to third-order functions can be solved exactly, requiring the function values at the ends and at the middle of the integration interval. The mid-interval value has the biggest influence on the result, and therefore, it is weighted with factor 4 [6]. The Integral is approximated by

$$I \approx \frac{\ell}{6} \left( f(x_1) + 4f(x_2) + f(x_3) \right)$$  \hspace{1cm} (2.4)

If the integral is of higher order than 3, or the function is not a polynomial, the solution is not exact any more. The numerical solution with Simpson’s rule for the presented integral is

$$I \approx \frac{2}{6} \left( f(-1) + 4f(0) + f(1) \right)$$

$$= \frac{2}{6} \left( \cos \left( \frac{-\pi}{2} \right) + 4 \cos(0) + \cos \left( \frac{\pi}{2} \right) \right) = 1.3333$$  \hspace{1cm} (2.5)

This means an error of 4.7%. It can be seen that the approximation with a quadratic element of a non-polynomial integral can provide big errors. The error can be minimized by splitting up the integration interval into subintervals. Thus, the effort in calculation rises.
2.4 Gaussian Quadrature

Simpson uses an equidistant distance between the integration points. In the Gaussian quadrature, the integration points have specified positions called Gauss points. These points are weighted with a weighting factor. The sum of the function values at these points multiplied by the weighting factors, is the approximated solution of the integral (see Equation (2.6)). The Gaussian quadrature provides an exact solution for integrals up to order $2S - 1$, with $S$ as the number of Gauss points [8]. Table 2.1 shows the Gauss points and weights for Gaussian quadrature with up to 5 points.

<table>
<thead>
<tr>
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<th>$\xi_s$</th>
<th>$w_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>$\pm \frac{1}{\sqrt{3}}$</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>$0$</td>
<td>$\frac{5}{9}$</td>
</tr>
<tr>
<td></td>
<td>$\pm \frac{\sqrt{3}}{3}$</td>
<td>$\frac{8}{9}$</td>
</tr>
<tr>
<td>4</td>
<td>$\pm 0.3399810436$</td>
<td>0.6521451549</td>
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<tr>
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<td>5</td>
<td>0.0000000000</td>
<td>0.5688888889</td>
</tr>
<tr>
<td></td>
<td>$\pm 0.5384693101$</td>
<td>0.4786286705</td>
</tr>
<tr>
<td></td>
<td>$\pm 0.9061798459$</td>
<td>0.2369268851</td>
</tr>
</tbody>
</table>

Table 2.1: Abscissas and weights for Gauss-Legendre quadrature

2.4.1 1D Gaussian Quadrature

Integrals over a function with one variable can be solved with one-dimensional Gaussian quadrature. If the integration goes from $-1$ to $1$, the procedure is called *standard Gaussian quadrature*. Integrals with different intervals can be transformed to standard Gaussian quadrature with a coordinate transformation.
The solution of the integral in Equation (2.1) with Gaussian quadrature with three Gauss points is presented in Table 2.2. A graphic output is shown in Figure 2.4.

![Figure 2.4: Numerical integration by Gaussian quadrature](image)

<table>
<thead>
<tr>
<th>$i$</th>
<th>$x_i$</th>
<th>$w_i$</th>
<th>$f(x_i)$</th>
<th>$f(x_i)w_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-\sqrt{\frac{3}{5}}\frac{5}{9}$</td>
<td>0.3467</td>
<td>0.1926</td>
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</tr>
<tr>
<td>2</td>
<td>0</td>
<td>$\frac{8}{9}$</td>
<td>1</td>
<td>0.8889</td>
</tr>
<tr>
<td>3</td>
<td>$\sqrt{\frac{3}{5}}\frac{5}{9}$</td>
<td>0.3467</td>
<td>0.1926</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\sum$</td>
<td></td>
<td>1.2741</td>
<td></td>
</tr>
</tbody>
</table>

Table 2.2: Gaussian quadrature

The error relative to the analytical solution is 0.07%. This method gives very accurate results compared to the effort in calculation, which is of enormous importance for an implementation in computer application software.

Only continuous functions can be integrated with Gaussian quadrature. Singularities mark discontinuities, so singular functions cannot be integrated in their normal form. They require special techniques, one of them is presented in Chapter 4.

### 2.4.2 2D Gaussian Quadrature

The Gaussian quadrature can also be used to solve double integrals (integrals over an area). The procedure is of the type "sum of sums". The integration area is covered with a grid, where the Gauss points define the positions of the rows and columns of the grid. Then, every node of a row is calculated and weighted with the weighting factors for the $\xi$- and $\eta$-direction. This is followed by the nodes of the next row. Finally, the sum of all results gives the solution.
\[ \int_{-1}^{1} \int_{-1}^{1} f(\xi, \eta) \, d\xi \, d\eta \approx \sum_{i=1}^{S_\xi} \sum_{j=1}^{S_\eta} f(\xi_i, \eta_j) w_i w_j \] (2.7)

Figure 2.5 shows the distribution of the integration nodes in a two-dimensional standard Gaussian quadrature of degree 3, Equation (2.7) represents the method.

Figure 2.5: Standard 2D Gaussian integration with local coordinates

An example for this method is shown in Section 4.3.

2.4.3 3D Gaussian Quadrature

Three-dimensional Gaussian quadrature is used to compute integrals over a volume. The procedure is similar as for area integrals.

\[ \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} f(\xi, \eta, \zeta) \, d\xi \, d\eta \, d\zeta \approx \sum_{i=1}^{S_\xi} \sum_{j=1}^{S_\eta} \sum_{k=1}^{S_\zeta} f(\xi_i, \eta_j, \zeta_k) w_i w_j w_k \] (2.8)
3 Fundamental Solutions

Fundamental solutions are essential for the BEM. They describe the reaction of an infinite homogenous domain due to a single load. There are of course different fundamental solutions for two-dimensional and three-dimensional problems. In the following only the fundamental solution for isotropic materials are discussed.

The derivation of the fundamental solutions is based on the condition of equilibrium [7, p. 85] and is quite complicated. The fundamental solution $U_{ij}$ describes the displacement field due to a unit load in $i$-direction in an infinite domain. The fact, that these fundamental solutions are exact solutions of the governing differential equation is one of the advantages of the BEM.

The deformation in x-direction ($j = x$) due to a unit load in x-direction ($i = x$) in a three-dimensional infinite domain is calculated, for example, by

$$U_{xx}(P, Q) = \frac{C}{r}(C_1 + r_x^2)$$

with

$$C = \frac{1}{16\pi G(1 - \nu)} \quad , \quad C_1 = 3 - 4\nu \ . \quad (3.1)$$

![Figure 3.1: 3D fundamental solution $U_{xx}$, $z = 0$, $\nu = 0.2$](image)
The basic variable is the distance $r$ between source point $P$ and field point $Q$. $r_x$ represents the projection of the vector $r$ on a straight line through source point $P$ and parallel to the $x$-axis. The distance $r_y$ is the projection parallel to the $y$-axis. The constants $G$ and $\nu$ are the shear modulus and the Poisson’s ratio respectively.

The graphic output of this function is a three-dimensional plot (see Figure 3.1). With this plot, the function is much easier to understand: The deformation tends to zero, as the field point $Q$ moves away from the load point $P$. The deformation near $P$ is large and gets larger as $Q$ approaches $P$. Actually, the deformation at $p$ is infinite. This is called a singularity. However, in nature the deformation will never be infinite because the area on which the force acts would have to be zero. Note, that the elevation of the surface in Figure 3.1 is not the deformation itself, but its amount in the specified direction.

Figure 3.2 shows the deformations in $y$-direction due to a point load in $x$-direction on a plane with $z = 0$. Naturally, the deformations are higher the more the field point $Q$ approaches to the load point $P$. Due to symmetry, the deformations are zero on the lines going through $P$ in direction of the load and rectangular to it.

![Figure 3.2: 3D fundamental solution $U_{xy}$, $z = 0$, $\nu = 0.2$](image)

Figure 3.3 shows the deformations on a plane with $z = 5$. There is no singularity in this plane, since the source point is not situated on this point.

The unit load causing this deformations acts ”under” this plane at $(0, 0, 0)$ in positive $x$-direction. The material on the negative $x$-half is attracted to the source point, the one on the positive half is pushed away by the load. As this plot describes the deformations in $z$-direction, the surface shows the real deformation of the observed plane, of course not to scale.

The isotropic fundamental solution $U_{ij}$ for a three-dimensional problem can be expressed as follows:
\[ U_{ij} = \frac{1}{16\pi G(1-\nu)r} \left( (3-4\nu)\delta_{ij} + r_i r_j \right) \]

with the Kronecker delta
\[ \delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \] (3.2)

and \( r_j = \frac{\partial r}{\partial x_j} \)

Some problems, such as straight dams and tunnels with constant conditions along their axis, are called plain strain problems. In a plane strain state, the strains normal to the \( x-y \)-plane are assumed to be zero \[5\]. These problems can be treated as two dimensional problems.

The deformations in \( j \)-direction due to a unit load in \( i \)-direction in a two-dimensional infinite domain can be calculated by

\[ U_{ij} = \frac{1}{8\pi G(1-\nu)} \left( (3-4\nu) \ln \frac{1}{r} \delta_{ij} + r_i r_j \right). \] (3.3)

Note, that the point load in the two-dimensional model represents a uniformly distributed load in the connected three-dimensional model.

**Figure 3.3**: 3D fundamental solution \( U_{xz}, z = 5, \nu = 0.2 \)

Numerical Integration - An Introduction for the Boundary Element Method
The tractions acting on a plane due to a unit load in an infinite domain are described by the fundamental solution $T_{ij}$, where $\theta$ corresponds to the angle between the vector $r$ and the outward normal $n$ of the plane. Equation (3.4) is the general form of the fundamental solution $T_{ij}$. The factors $\alpha$ and $\beta$ are different for two- and three-dimensional problems and are shown in Table 3.1.

$$T_{ij} = \frac{-1}{4\alpha\pi(1-\nu)r^2} \left( ((1-2\nu)\delta_{ij} + \beta r_i r_j) \cos \theta - (1-2\nu)(n_j r_i - n_i r_j) \right)$$  \hspace{1cm} (3.4)

<table>
<thead>
<tr>
<th>Coeff.</th>
<th>2-D</th>
<th>3-D</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
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<td>2</td>
</tr>
<tr>
<td>$\beta$</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 3.1: Coefficients for two- and three-dimensional fundamental solutions

Figure 3.4 is a plot of the fundamental solution $T_{yx}$. It describes the tractions at any point on a level with $z = 0$ due to a unit point load in $y$-direction. The outward normal $n$ points in $x$-direction. Note that $T_{ij} \neq T_{ji}$ with unchanged outward normal. The tractions are antimetric to the $x$-axis and symmetric to the $y$-axis. The sum of all tractions in a section through the domain has to be zero (condition of equilibrium).

Figure 3.4: 3D fundamental solution $T_{yx}, z = 0, \nu = 0.2, n = [1, 0, 0]$
The fundamental solution $E_{ijk}$ describes the corresponding strain field in a homogenous infinite domain [10]. Here, the equation depends on three indices: two for the strain $\varepsilon_{jk}$ and one for the direction of the load.

$$E_{ijk} = -\frac{1}{8\pi G(1 - \nu)r^2} \left((1 - 2\nu)(r_j \delta_{ij} + r_i \delta_{jk}) - r_i \delta_{jk} + \beta r_j r_i r_k\right)$$ (3.5)

Figure 3.5 shows a plot of the fundamental solution $E_{yxy}$. It shows the strain $\varepsilon_{xy}$ due to a force in $y$-direction.

Figure 3.5: 3D fundamental solution $E_{yxy}, z = 0, \nu = 0.2$

More fundamental solutions for elastostatics can be found for example in Thoeni [10].
4 Lachat-Watson Transformation

The integration of singular functions are an important issue in the BEM. Most numerical methods are not able to deal with singularities, therefore, special techniques have to be used. Lachat and Watson [9] for example introduced a technique to solve weakly singular integrals over quadrilateral boundary elements for three-dimensional problems. The Lachat-Watson transformation can be used for boundary integrals for three-dimensional problems and for domain integrals (see Section 1.2.7) for two- and three-dimensional problems.

4.1 2D Transformation

If the singular point is located at a corner of a quadrilateral, this parent element is split up into two triangular subelements (see Figure 4.1). Otherwise, if the singular point is located on a mid-node of a quadratic element this element is split up into three subelements. Now, if the integration is done over the subelements, and the singularity cancels out, because the Jacobian of this transformation is equal to zero in the point of singularity.

E.g. the integral which has to be evaluated contains the singularity of the type 1/r and has the form

\[ \int_{\Gamma} U_{ij}(P, Q) N_\alpha(Q) \, d\Gamma = \int_{-1}^{1} \int_{-1}^{1} U_{ij}(P, Q(\xi, \eta)) N_\alpha(\xi, \eta) J(\xi, \eta) \, d\xi \, d\eta, \] (4.1)

where Equation (3.2) applies for the fundamental solution.

If point P is not on a corner of the element, but inside it, the parent element has to be divided into four triangular subelements (see Figure 4.2). For each subelement, an own transformation has to be build. This would be inconvenient for programming, thus a more general transformation formula was developed in this work.
\[ \xi^{(s)} = \bar{N}_1 v_1(s) + \bar{N}_2 v_2(s) + \bar{N}_3 \xi_p, \]
\[ \eta^{(s)} = -\bar{N}_1 v_2(s) + \bar{N}_2 v_1(s) + \bar{N}_3 \eta_p, \]

(4.2)

where the coefficients \( v_1(s) \) and \( v_2(s) \) are defined in Table 4.1.

<table>
<thead>
<tr>
<th>s</th>
<th>( v_1(s) )</th>
<th>( v_2(s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>4</td>
<td>-1</td>
<td>-1</td>
</tr>
</tbody>
</table>

Table 4.1: Coefficients for two-dimensional transformation of the triangular subelements

The shape functions of a triangular subelement are given by

\[ \bar{N}_1 = \frac{1}{4} (1 + \xi)(1 - \eta) \]
\[ \bar{N}_2 = \frac{1}{4} (1 + \xi)(1 + \eta) \]
\[ \bar{N}_3 = \frac{1}{2} (1 - \xi) \]

(4.3)

where \( \xi \) and \( \eta \) correspond to the local coordinates of the subelement.

\[ \int_{\Gamma^e} U_{ij}(P, Q) N_\alpha(Q) \, d\Gamma = \sum_{se=1}^{se} \int_{-1}^{1} \int_{-1}^{1} U_{ij}(P, Q(\bar{\xi}, \bar{\eta})) N_\alpha(\bar{\xi}, \bar{\eta}) f(\bar{\xi}, \bar{\eta}) \bar{J}_{se}(\bar{\xi}, \bar{\eta}) \, d\bar{\xi} \, d\bar{\eta}. \]

(4.4)
The modified Jacobian is given by

\[
\bar{J}_{se}(\bar{\xi}, \bar{\eta}) = \frac{\partial \xi}{\partial \bar{\xi}} \frac{\partial \eta}{\partial \bar{\eta}} - \frac{\partial \eta}{\partial \bar{\xi}} \frac{\partial \xi}{\partial \bar{\eta}}.
\]  

(4.5)

In the example in Section 4.3 can be seen that Equation (4.5) becomes zero if \(\bar{\xi} = -1\), and therefore, the singularity cancels out and all integrals of Equation (4.4) can be integrated by using standard Gaussian quadrature.

The transformation algorithm is designed for all of the four possible subelements. If point \(P\) would lie on a corner of the parent element, two subelements would degenerate to a point. It would be an unnecessary effort to integrate these subelements, as the result is zero anyway. In consequence, skip conditions are implemented in the algorithm:

- if \(\eta_p = -1\) skip subelement 1,
- if \(\xi_p = 1\) skip subelement 2,
- if \(\eta_p = 1\) skip subelement 3,
- if \(\xi_p = -1\) skip subelement 4.

### 4.2 3D Transformation

Domain integrals become singular, if the source point lies on the cell to be integrated. Weak singularities of two-dimensional problems can be solved with the Lachat-Watson transformation. This technique works also for three-dimensional problems. There, the parent cell is divided in up to six pyramids, depending on whether point \(P\) lies on a corner, on an edge, on the surface or inside the cell.

E.g., the integral which has to be evaluated contains the singularity of the type \(1/r^2\) and has the form

\[
\int_{\Omega} E_{ijk} (P,q) N_a(q) \, d\Omega = 
\int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} E_{ijk}(P,q(\xi,\eta,\zeta)) N_a(\xi,\eta,\zeta) J(\xi,\eta,\zeta) \, d\xi \, d\eta \, d\zeta.
\]  

(4.6)

The shape functions for a pyramidal subelement are given by

\[
\bar{N}_1 = \frac{1}{8} (1 + \bar{\xi})(1 + \bar{\eta})(1 + \bar{\zeta})
\]

\[
\bar{N}_2 = \frac{1}{8} (1 + \bar{\xi})(1 - \bar{\eta})(1 + \bar{\zeta})
\]

\[
\bar{N}_3 = \frac{1}{8} (1 + \bar{\xi})(1 - \bar{\eta})(1 - \bar{\zeta})
\]

\[
\bar{N}_4 = \frac{1}{8} (1 + \bar{\xi})(1 + \bar{\eta})(1 - \bar{\zeta})
\]

\[
\bar{N}_5 = \frac{1}{2} (1 - \bar{\xi}).
\]  

(4.7)

The following equations are the basis of a general coordinate transformation for the three-dimensional cell subdivision:

\[
\xi^{(s)} = \bar{N}_1 v_1(s) + \bar{N}_2 v_2(s) + \bar{N}_3 v_3(s) + \bar{N}_4 v_4(s) + \bar{N}_5 \xi_p
\]

\[
\eta^{(s)} = \bar{N}_1 v_3(s) + \bar{N}_2 v_4(s) + \bar{N}_3 v_5(s) + \bar{N}_4 v_6(s) + \bar{N}_5 \eta_p
\]

\[
\zeta^{(s)} = \bar{N}_1 v_7(s) + \bar{N}_2 v_8(s) + \bar{N}_3 v_9(s) + \bar{N}_4 v_8(s) + \bar{N}_5 \zeta_p.
\]  

(4.8)
Table 4.2: Coefficients for three-dimensional transformation functions

<table>
<thead>
<tr>
<th>s</th>
<th>v_1(s)</th>
<th>v_2(s)</th>
<th>v_3(s)</th>
<th>v_4(s)</th>
<th>v_5(s)</th>
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<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>6</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Figure 4.3: General subdivision for a three-dimensional cell
where the coefficients $v_1(s) - v_8(s)$ are defined in Table 4.2.

Equation (4.6) can now be written as a sum of the integrals over the pyramidal subelements:

\[
\int_{\Omega} \mathbf{E}_{ijk}(P, q) N_\alpha(q) \, d\Omega = \sum_{s=1}^{L^se} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \mathbf{E}_{ijk}(P, q(\xi, \eta, \zeta)) N_\alpha(\xi, \eta, \zeta) J_{se}(\xi, \eta, \zeta) \, d\xi \, d\eta \, d\zeta ,
\]

(4.9)

where the modified Jacobian is given by

\[
J_{se}(\xi, \eta, \zeta) = \frac{\partial \xi}{\partial \xi} \frac{\partial \eta}{\partial \eta} \frac{\partial \zeta}{\partial \zeta} + \frac{\partial \xi}{\partial \eta} \frac{\partial \eta}{\partial \zeta} \frac{\partial \zeta}{\partial \xi} + \frac{\partial \xi}{\partial \zeta} \frac{\partial \eta}{\partial \xi} \frac{\partial \zeta}{\partial \eta} - \left( \frac{\partial \xi}{\partial \zeta} \frac{\partial \eta}{\partial \eta} \frac{\partial \zeta}{\partial \xi} + \frac{\partial \xi}{\partial \eta} \frac{\partial \eta}{\partial \zeta} \frac{\partial \zeta}{\partial \eta} + \frac{\partial \xi}{\partial \xi} \frac{\partial \eta}{\partial \zeta} \frac{\partial \zeta}{\partial \eta} \right).
\]

(4.10)

Equation (4.10) becomes again zero if $\xi = -1$, and therefore, the singularity cancels out and all integrals of Equation (4.9) can be integrated by using standard Gaussian quadrature.

In general six subelements have to be considered, and therefore, $L^se = 6$. However, if the point of singularity lies on the outer boundary of the cell under consideration, only three or four subelements have to be considered. The conditions to skip a pyramidal subelement are similar to the one for two-dimensional problems:

- if $\eta_p = -1$ skip subelement 1,
- if $\xi_p = 1$ skip subelement 2,
- if $\eta_p = 1$ skip subelement 3,
- if $\xi_p = -1$ skip subelement 4,
- if $\zeta_p = -1$ skip subelement 5,
- if $\zeta_p = 1$ skip subelement 6.

### 4.3 Example

In the following, an example is provided, that compares an analytical solution of the integrals over the most important parts of a weakly singular fundamental solution with a numerical solution. The Lachat-Watson transformation is used to compute the numerical solution.

The function to be integrated is

\[
f(x, y) = \frac{(x + 1)^3}{r^3} \frac{1}{r}.
\]

(4.11)

The element to be integrated is shown in Figure 4.4.

The coordinates of the nodes of the element are chosen in such a way, that they correspond to the intrinsic coordinates of the element. Therefore, no transformation from global to local coordinates is needed. The coordinates of the point of singularity $P$ are $P(-1.0, -1.0)$. 

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The analytical solution of the integral is

\[ \int \int f(x, y) = \int_{-1}^{1} \int_{-1}^{1} \frac{(x + 1)^3}{r^4} \, dx \, dy \]

\[ = \int_{-1}^{1} \int_{-1}^{1} \frac{x^3}{((x + 1)^2 + (y + 1)^2)^2} \, dx \, dy \]

\[ = \int_{-1}^{1} \left[ \frac{y^2}{2(y^2 + x^2)} \right] + \frac{1}{2} \ln|y^2 + x^2| \right] \, dy \]

\[ = \frac{1}{2} \int_{-1}^{1} \frac{y^2}{y^2 + 2} \, dy + \frac{1}{2} \int_{-1}^{1} \ln|y^2 + 2| \, dy - \frac{1}{2} \int_{-1}^{1} \frac{y^2}{y^2} \, dy - \frac{1}{2} \int_{-1}^{1} \ln|y^2| \, dy \]

\[ = \frac{1}{2} \left[ y + 2 \arctan \frac{y}{2} \right]_{-1}^{1} + \frac{1}{2} \left[ 4 \arctan \frac{y}{2} + y(\ln(y^2 + 4) - 2) \right]_{-1}^{1} \]

\[ - \frac{1}{2} \left[ y \right]_{-1}^{1} - \frac{1}{2} \left[ y(\ln(y^2) - 2) \right]_{-1}^{1} \]

\[ = \frac{1}{2}(2 - 2\arctan \frac{y}{2}) + \frac{1}{2}(\pi + 0.158883) - 1 - (\ln(4) - 2) \]

\[ = \frac{\pi}{4} + \ln 2 \approx 1.47855 \, . \]

For computing the numerical solution the parent element is split up into four subelements, according to the procedure indicated in Section 4.1. Subelement 1 and 4 degenerate to lines and they do not have to be considered any longer in this calculation. The remaining subelements have new intrinsic coordinates \( \xi \) and \( \eta \) which are calculated according to Equation 4.2.

The transformation for subelement 2 is given by

\[ x_2 = \frac{1}{4}(1 + \xi)(1 - \eta)p_1(2) + \frac{1}{4}(1 + \xi)(1 + \eta)p_2(2) + \frac{1}{4}(1 - \xi)p_\rho \]

\[ = \frac{1}{4}(1 + \xi)(1 - \eta)1 + \frac{1}{4}(1 + \xi)(1 + \eta)1 + \frac{1}{4}(1 - \xi)(-1) = \xi \]

\[ y_2 = -\frac{1}{4}(1 + \xi)(1 - \eta)p_2(2) + \frac{1}{4}(1 + \xi)(1 + \eta)p_1(2) + \frac{1}{4}(1 - \xi)p_\rho \]

\[ = -\frac{1}{4}(1 + \xi)(1 - \eta)1 + \frac{1}{4}(1 + \xi)(1 + \eta)1 + \frac{1}{4}(1 - \xi)(-1) \]

\[ = \frac{1}{4}(1 + \xi)\eta - \frac{1}{4}(1 - \xi) \, . \]

The modified Jacobian is defined in Equation (4.5) and becomes \( J_2 = \frac{1}{2}(\xi + 1) \).
The transformation for subelement 3 is given by

\[ x_3 = \frac{1}{2}(1 + \bar{\xi})(1 - \bar{\eta})v_1(3) + \frac{1}{2}(1 + \bar{\xi})(1 + \bar{\eta})v_2(3) + \frac{1}{2}(1 - \bar{\xi})\xi_p \]

\[ y_3 = -\frac{1}{4}(1 + \bar{\xi})(1 - \bar{\eta})v_2(3) + \frac{1}{4}(1 + \bar{\xi})(1 + \bar{\eta})v_1(3) + \frac{1}{2}(1 - \bar{\xi})\xi_p \]

(4.14)

The modified Jacobian becomes \( \bar{J}_3 = \frac{1}{2}(\bar{\xi} + 1) \).

By introducing this transformation, the integration is done numerically with standard two-dimensional Gauss quadrature over subelements 2 and 3. The number of Gauss points is set equal to 3 in both directions.

For the numerical solution, the Gauss points of the element given in \( \bar{\xi}, \bar{\eta} \) coordinates are transformed in Cartesian coordinates and inserted in the function \( f(x, y) \). In this transformation the modified Jacobian \( \bar{J}_{Se} \) has to be considered. The calculation of the numerical integration over subelement 2 is shown in table 4.3.

<table>
<thead>
<tr>
<th>( \xi_i )</th>
<th>( \bar{\xi}_i )</th>
<th>( \eta_j )</th>
<th>( \bar{\eta}_j )</th>
<th>( x_i )</th>
<th>( y_i )</th>
<th>( f(x_i, y_i) )</th>
<th>( \bar{J}(\bar{\xi}_i, \bar{\eta}_j) )</th>
<th>( \bar{J}w_i w_j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(-\sqrt{\frac{3}{5}})</td>
<td>(-\sqrt{\frac{3}{5}})</td>
<td>(-\sqrt{\frac{3}{5}})</td>
<td>(-0.7746)</td>
<td>(-0.9746)</td>
<td>4.3260</td>
<td>0.1127</td>
<td>0.1505</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>(-\sqrt{\frac{3}{5}})</td>
<td>(-0.7746)</td>
<td>(-0.8733)</td>
<td>2.8394</td>
<td>0.1127</td>
<td>0.1580</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>(-\sqrt{\frac{3}{5}})</td>
<td>(-\sqrt{\frac{3}{5}})</td>
<td>(-0.7746)</td>
<td>0.2000</td>
<td>(-0.8000)</td>
<td>0.1127</td>
<td>0.0483</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>(-\sqrt{\frac{3}{5}})</td>
<td>0.0000</td>
<td>(-0.8873)</td>
<td>0.9751</td>
<td>0.5000</td>
<td>0.2408</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>(-\sqrt{\frac{3}{5}})</td>
<td>(-\sqrt{\frac{3}{5}})</td>
<td>0.0000</td>
<td>(-0.5000)</td>
<td>0.6400</td>
<td>0.5000</td>
<td>0.2528</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>(-\sqrt{\frac{3}{5}})</td>
<td>(-\sqrt{\frac{3}{5}})</td>
<td>0.0000</td>
<td>(-0.1127)</td>
<td>0.3130</td>
<td>0.5000</td>
<td>0.0773</td>
<td></td>
</tr>
</tbody>
</table>

\[ \sum = 1.2845 \]

**Table 4.3:** Numerical integration of the lower subelement
The same procedure can be used to evaluate the integration over subelement 3.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\xi_i$</th>
<th>$\eta_i$</th>
<th>$\bar{w}_i$</th>
<th>$\bar{w}_j$</th>
<th>$x_i$</th>
<th>$y_i$</th>
<th>$f(x_i, y_i)$</th>
<th>$\bar{f}(\xi_i, \eta_j)$</th>
<th>$f\bar{w}_i \bar{w}_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-\sqrt{3}/4$</td>
<td>$1$</td>
<td>$-\sqrt{3}/4$</td>
<td>$\frac{5}{6}$</td>
<td>$-0.8000$</td>
<td>$-0.7746$</td>
<td>$0.9702$</td>
<td>$0.1127$</td>
<td>$0.0337$</td>
</tr>
<tr>
<td>2</td>
<td>$0$</td>
<td>$0$</td>
<td>$\frac{5}{6}$</td>
<td>$\frac{5}{6}$</td>
<td>$-0.8873$</td>
<td>$-0.7746$</td>
<td>$0.0062$</td>
<td>$0.1127$</td>
<td>$0.0198$</td>
</tr>
<tr>
<td>3</td>
<td>$\sqrt{3}/5$</td>
<td>$\frac{5}{6}$</td>
<td>$\frac{5}{6}$</td>
<td>$\frac{5}{6}$</td>
<td>$-0.9746$</td>
<td>$-0.7746$</td>
<td>$0.0062$</td>
<td>$0.11270$</td>
<td>$0.0002$</td>
</tr>
<tr>
<td>2</td>
<td>$0$</td>
<td>$\frac{5}{6}$</td>
<td>$\frac{5}{6}$</td>
<td>$\frac{5}{6}$</td>
<td>$-0.5000$</td>
<td>$0.0000$</td>
<td>$0.0800$</td>
<td>$0.5000$</td>
<td>$0.0316$</td>
</tr>
<tr>
<td>3</td>
<td>$\sqrt{3}/5$</td>
<td>$\frac{5}{6}$</td>
<td>$\frac{5}{6}$</td>
<td>$\frac{5}{6}$</td>
<td>$-0.8873$</td>
<td>$0.0000$</td>
<td>$0.0014$</td>
<td>$0.5000$</td>
<td>$0.0003$</td>
</tr>
<tr>
<td>3</td>
<td>$\sqrt{3}/5$</td>
<td>$\frac{5}{6}$</td>
<td>$\frac{5}{6}$</td>
<td>$\frac{5}{6}$</td>
<td>$0.5746$</td>
<td>$0.7746$</td>
<td>$0.1232$</td>
<td>$0.8873$</td>
<td>$0.0337$</td>
</tr>
<tr>
<td>2</td>
<td>$0$</td>
<td>$\frac{5}{6}$</td>
<td>$\frac{5}{6}$</td>
<td>$\frac{5}{6}$</td>
<td>$-0.1127$</td>
<td>$0.7746$</td>
<td>$0.0451$</td>
<td>$0.8873$</td>
<td>$0.0198$</td>
</tr>
<tr>
<td>3</td>
<td>$\sqrt{3}/5$</td>
<td>$\frac{5}{6}$</td>
<td>$\frac{5}{6}$</td>
<td>$\frac{5}{6}$</td>
<td>$-0.8000$</td>
<td>$0.7746$</td>
<td>$0.0008$</td>
<td>$0.8873$</td>
<td>$0.0002$</td>
</tr>
</tbody>
</table>

$\sum = 0.1934$

**Table 4.4:** Numerical integration of the upper subelement

The sum of both subelements is 1.4779. This means that the numerical integration gives an error of 0.04%. However, by using more Gauss points the numerical solution can be improved.
5 Graphic Output Code

The plots of the fundamental solutions in Chapter 3 where generated in MATLAB. In the following, the code for the two-dimensional and three-dimensional displacement fundamental solution is presented.

```matlab
% Fundamental Solution Uij in 2D

clear all
warning off all
G=10000; % Material parameters
nu=0.2;
n=150; % Mesh parameters
m=n/15;
d=2; % 2D

phi=(0:m*pi/n:2*pi); % set upper border to pi for a half circle
rn=0:m:n;

X=zeros(length(rn), length(phi)); % generation of matrices for the mesh
Y=zeros(length(rn), length(phi));

for i=1:length(rn)
    for j=1:length(phi)
        X(i,j)=rn(i)*cos(phi(j)); % write matrices for polar coordinates
        Y(i,j)=rn(i)*sin(phi(j));
    end
end

r=sqrt(X.^2+Y.^2);

A=cell(2,1); % cell array with r,x and r,y
A{1}=X./r;
A{2}=Y./r;

C2=1/(8*G*pi*(1-nu)); % constants
C3=3-4*nu;
D=diag(ones(1,2),0); % matrix for Kronecker delta

p=1; % position of plot in plot window

for i=1:d % i and j stand for the indices of the fundamental solution
    for j=1:d
        U=C2*(C3*log(r)+D(i,j)+A{i}.*A{j}); % --- KERNEL ---

        figure(p); surf(X,Y,U) % plot
        alpha(0.7) % plot properties
        axis([-n n -n n])
        view([-15, -15, 10])
        xlabel('X-axis')
        ylabel('Y-axis')
        title(['U', num2str(i), num2str(j)])

        p=p+1;
    end
end

warning on all
```

clear all
warning off all
G=10000; \% Material parameters
nu=0.2;
n=150; \% Mesh parameters
m=n/15;
d=3; \% 3D

phi=(0:m*pi/n:2*pi); \% set upper border to pi for a half circle
rn=0:m:n;
X=zeros(length(rn), length(phi)); \% generation of matrices for the mesh
X=zeros(length(rn), length(phi));

for i=1:length(rn)
    for j=1:length(phi)
        X(i,j)=rn(i)*cos(phi(j)); \% write matrices for polar coordinates
        Y(i,j)=rn(i)*sin(phi(j));
    end
end

Z=input('section plane Z=?');
if isempty(d)
    Z=0
end
r=sqrt(X.^2+Y.^2+Z.^2);

A=cell(3,1); \% cell array with r,x and r,y and r,z
A{1}=X./r;
A{2}=Y./r;
A{3}=Z./r;

C2=1/(16*pi*G*(1-nu)); \% constants
C3=3-4*nu;
D=diag(ones(1,3),0); \% matrix for Kronecker delta

p=1; \% position of plot in plot window

for i=1:d \% i and j stand for the indices of the fundamental solution
    for j=1:d
        U=C2./r.*(C3.*D(i,j)*A{i}.*A{j}); \% --- KERNEL ---

        figure(p); surf(X,Y,U) \% plot
        alpha(0.7) \% plot properties
        axis([-n n -n n])
        view([-15, -15, 10])
        xlabel('X-axis')
        ylabel('Y-axis')
        title(['U', num2str(i), num2str(j)])

        p=p+1;
    end
end

warning on all
The fundamental solution \( W_{ijkl} \) is the most complex fundamental solution. In this case, the dimension of the problem can be chosen in the program.

```matlab
% Fundamental Solution Wijkl

clear all
warning off all
G=10000; % Material parameters
nu=0.2;
n=150; % Mesh parameters
m=n/15;

d=input('geometric_dimension_(Z/3)?_');
if isempty(d)
d=2
end

phi=(0:m*pi/n:2*pi); % set upper border to pi for a half circle
rn=m:n;

X=zeros(length(rn), length(phi)); % generation of matrices for the mesh
Y=zeros(length(rn), length(phi));

for i=1:length(rn)
    for j=1:length(phi)
        X(i,j)=rn(i)*cos(phi(j)); % write matrices for polar coordinates
        Y(i,j)=rn(i)*sin(phi(j));
    end
end

if (d==2) % two-dimensional problem
    alph=1;
    beta=2;
    gamma=4;
    r=sqrt(X.^2+Y.^2);
    A=cell(2,1); % cell array with r,x and r,y
    A{1}=X./r;
    A{2}=Y./r;
elseif (d==3) % three-dimensional problem
    alph=2;
    beta=3;
    gamma=5;
    Z=input('section_plane_Z=?_');
    if isempty(d)
        Z=0
    end
    r=sqrt(X.^2+Y.^2+Z.^2);
    A=cell(3,1); % cell array with r,x and r,y and r,z
    A{1}=X./r;
    A{2}=Y./r;
    A{3}=Z./r;
else
    stop
end

C2=1/(4*alph*pi*(1-nu)); % constants
C3=1-2*nu;
p=1; % position of plot in plot window
D=diag(ones(1,3),0);
```
for i=1:d  % i, j, k and l stand for the indices of the fundamental solution
   for j=1:d
      for k=1:d
         for l=1:d
            W=C2./r.*(beta.*D(i,k).*D(l,j)+D(j,k).*D(l,i).*D(i,j).*D(k,l)
            + (beta*D(i,j)).*A(k).*A(l)+beta.*nu.*D(l,i).*A(1).*A(k)
            - D(k,l).*A(1).*A(1).*A(1).*A(1).*A(1).*A(1).*A(1))
            + beta*D(k,l).*A(1).*A(1).*A(1).*A(1).*A(1).*A(1).*A(1).*A(1));  % --- KERNEL ---

            figure(p); surf(X,Y,W)  % plot
            axis([-n n -n n])
            xlabel('X-axis')
            ylabel('Y-axis')
            title(['W', num2str(i), num2str(j), num2str(k), num2str(l)])
            p=p+1;
         end
      end
   end
end
warning on all
Bibliography


